

Remarks on the nonvanishing of cohomology groups for perverse sheaves on abelian varieties

Rainer Weissauer

Let X be an abelian variety over an algebraically closed field k of dimension g and let K be an irreducible perverse sheaf in $D_c^b(X, \Lambda)$ for $\Lambda = \overline{\mathbb{Q}}_\ell$. If the base field k has positive characteristic, we assume that K is defined over a field that is finitely generated over its prime field with ℓ different from the characteristic. Suppose that not all cohomology groups $H^\nu(X, K)$ are zero and let denote $d(K) = \max\{\nu \mid H^\nu(X, K) \neq 0\}$. Notice $d(K) \geq 0$, by the Hard Lefschetz Theorem.

Theorem. *For $d = d(K) > 0$ we have*

$$\dim_\Lambda(H^{d-1}(X, K)) > 2d/(d+g) \cdot \dim_\Lambda(H^d(X, K)) .$$

If furthermore X is a simple abelian variety, then $\dim_\Lambda(H^{d-1}(X, K)) > d \cdot \dim_\Lambda(H^d(X, K))$.

Remark. By the Hard Lefschetz Theorem an immediate consequence of this theorem is the assertion: $H^\nu(X, K) \neq 0$ if and only if $\nu \in [-d(K), d(K)]$. So for character twists K_χ [KrW] the sets $V_i(K) = \{\chi \mid H^i(X, K_\chi) \neq 0\}$ satisfy $V_{i+1}(K) \subseteq V_i(K)$ for all $i \geq 0$. For an arbitrary projective smooth variety Y over k with Albanese morphism $f : Y \rightarrow X$ and a perverse sheaf L on Y the decomposition theorem gives $Rf_*(L) \cong \bigoplus {}^pH^i(Rf_*(L))[-i]$ and $H^\nu(Y, L) \cong \bigoplus_{j+i=\nu} H^j(X, {}^pH^i(Rf_*(L)))$. From the relative Hard Lefschetz Theorem and the theorem above applied to the irreducible constituents K of the semisimple perverse cohomology sheaves ${}^pH^i(Rf_*(L))$ we therefore obtain

Corollary 1. *Let L be an irreducible perverse sheaf L on a smooth projective variety Y with $d = \max\{\nu \mid H^\nu(Y, L) \neq 0\}$. Suppose the Albanese morphism $f : Y \rightarrow X$ is not trivial and suppose $H^d(Y, L) \neq H^0(X, {}^pH^d(Rf_*(L)))$ (e.g. this is the case if the fibers of f have dimension $< d$). Then $H^\nu(Y, L) \neq 0$ holds if and only if $\nu \in [-d, d]$.*

Proof of the theorem. First suppose that K is negligible, i.e. of the form $K \cong \pi^*(Q)[q]$ for a perverse sheaf Q on a quotient abelian variety $\pi : X \rightarrow X/A$ defined by an abelian subvariety $A \subseteq X$ of dimension $q > 0$. Then $d = d(K) = d(Q) + q$ since $H^\bullet(X, K) \cong \bigoplus_{i=0}^{2q} \binom{2q}{i} \cdot H^\bullet(X/A, Q[i+q])$. Hence $H^d(X, K) \cong H^{d(Q)}(X/A, Q)$ and $H^{d-1}(X, K) \cong 2q \cdot H^{d(Q)}(X/A, Q) \oplus H^{d(Q)-1}(X/A, Q)$. Since $2q > 2d/(d+g)$, our claim follows in this case; similarly $2q = 2g > d$ in the case where $X = A$ is simple. Therefore we now make the

Assumption. *Suppose K is irreducible, but not negligible. Furthermore suppose $d > 0$.*

For the perverse sheaf K on X consider the Laurent polynomial $h_t(X, K) = \sum_\nu a_\nu t^\nu$ defined by $a_\nu = \dim_\Lambda(H^\nu(X, K))$. Then $d = d(K)$ is the largest integer ν such that $a_\nu \neq 0$.

Choose an integer r minimal such that $r \cdot d > g$. Hence $r > 1$ and $r \cdot d < g + d$. The r -th convolution power of K is a direct sum of a perverse sheaf K_r on X and a finite direct sum of complexes $L_\mu[n_\mu]$ with negligible perverse sheaves L_μ on X of the form:

- $L_\mu = \pi_\mu^*(Q_\mu)[g_\mu]$ for irreducible not negligible perverse sheaves Q_μ on X/A_μ
- $\pi_\mu : X \rightarrow X/A_\mu$ is the quotient by an abelian subvariety A_μ of X of dimension $g_\mu > 0$.

This follows from [KrW], [W], and for this assertion we have to assume that the perverse sheaf K is defined over a finitely generated field over the prime field in the case of positive characteristic [W].

Then $h_t(X, L_\mu[n_\mu]) = \sum_\nu \dim(H^\nu(X, L_\mu[n_\mu]) \cdot t^\nu = \sum_{\nu \leq d_\mu} b_{\mu\nu} t^\nu$ for integers $b_{\mu\nu} \geq 0$, and we may assume $b_\mu = b_{\mu d_\mu} \geq 1$ since we can ignore cohomologically trivial summands in the following. Let T denote the set of all indices μ such that $d_\mu + g_\mu = r \cdot d$ holds. By well known cohomological bounds [BBD], the cohomology of an irreducible perverse sheaf on X vanishes in degrees $\geq g$ unless it is negligible. Since $r \cdot d \geq g$, the Künneth formula in the form $H^\bullet(X, K^{*r}) \cong H^\bullet(X, K)^{\otimes r}$ and a comparison of coefficients at t^{rd} implies

$$(a_d)^r = \sum_{\mu \in T} b_\mu.$$

Similarly, now using $r \cdot d \geq g + 1$, by comparing coefficients at t^{rd-1} we obtain

$$r \cdot a_{d-1} (a_d)^{r-1} \geq \sum_{\mu \in T} 2g_\mu b_\mu \geq 2 \cdot \min_\mu (g_\mu) \cdot (a_d)^r.$$

Indeed, the second equality follows from the formula $\sum_{\mu \in T} b_\mu = (a_d)^r$ above. For the first inequality we exploited the fact that all coefficients $b_{\mu\nu}$ in $h_t(X, L_\mu[n_\mu]) = (t + 2 + t^{-1})^{g_\mu} \cdot h_t(X/A_\mu, Q_\mu[n_\mu]) = (t^{g_\mu} + 2g_\mu t^{g_\mu-1} + \dots)(b_\mu t^{d_\mu} + \dots)$ are nonnegative. We conclude

$$a_{d-1} \geq \frac{2 \min_\mu (g_\mu)}{r} \cdot a_d \geq \frac{2}{r} \cdot a_d > \frac{2d}{g+d} \cdot a_d$$

where the last inequality follows from $r \cdot d < g + d$. If X is simple, then $\min_\mu (g_\mu) = g$ and hence $a_{d-1} \geq \frac{2g}{r} a_d$. Now $r \cdot d < g + d < 2g$ implies $a_{d-1} > d \cdot a_d$. QED

Remark. $d(K)$ for the intersection cohomology sheaf K of an irreducible subvariety Y of X is the dimension of Y . In this case there exist stronger geometric estimates than those from the theorem above. However, already when Y is a variety of maximal Albanese dimension and K is an arbitrary irreducible constituent of the direct image of the intersection cohomology perverse sheaf on Y under the Albanese morphism $f : Y \rightarrow X = Alb(Y)$ I am not aware of estimates of the above form in the literature.

Next, consider a finite Galois morphism

$$\pi : \tilde{Y} \rightarrow Y$$

between smooth complex varieties of dimension n with Galois group Γ , where we view Γ to act on \tilde{Y} from the right. For every isomorphism class ϕ of irreducible representations V_ϕ of Γ let $m_\nu(\phi)$ denote the multiplicity of the irreducible representation ϕ of Γ on $H^{\nu+n}(\tilde{Y}, \mathbb{C})$.

For simplicity, from now on suppose that Y is projective and $f : Y \rightarrow Alb(Y) = X$ is a closed embedding. Then the theorem above implies

Corollary 2. *If $d = d(K_\phi) > 0$, then $m_{d-1}(\phi) > 2dm_d(\phi)/(d+g) > 0$.*

Proof. For every class ϕ there exists an irreducible perverse sheaf K_ϕ on Y and a Γ -equivariant isomorphism $H^{\bullet+n}(\tilde{Y}, \mathbb{C}) \cong \bigoplus_\phi V_\phi \otimes_{\mathbb{C}} H^\bullet(Y, K_\phi)$, where Γ acts on V_ϕ by ϕ and trivially on $H^\bullet(Y, K_\phi)$. For unramified π , this immediately follows from [KiW], remark 15.3 (d). Applying this remark for the restriction of π to $\pi^{-1}(U)$, for the open dense subset $U \subseteq Y$ obtained by removing the ramification divisor of π , by perverse analytic continuation in general it suffices to observe that for $\delta_{\tilde{Y}} = \mathbb{C}_{\tilde{Y}}[n]$ the semisimple perverse sheaf $\pi_*(\delta_{\tilde{Y}})$ on Y has irreducible perverse constituents K whose restriction to U are nontrivial. To show this notice that $\text{Hom}(\pi_*(\delta_{\tilde{Y}}), K) = \text{Hom}(\delta_{\tilde{Y}}, \pi^!(K))$ vanishes if K (and hence $\pi^!(K)$) is a perverse sheaf with support of dimension $< \dim(Y)$. Indeed, since $\delta_{\tilde{Y}}$ is an irreducible perverse sheaf with support of dimension $\dim(Y)$, $\text{Hom}(\delta_{\tilde{Y}}, \pi^!(K))$ is zero. This being said, we obtain $m_\nu(\phi) = \dim(H^\nu(Y, K_\phi))$. Since f is a closed immersion, the direct images of K_ϕ under the Albanese morphism again are irreducible perverse sheaves. So we can apply the theorem. QED

Still suppose $\pi : \tilde{Y} \rightarrow Y$ is a Galois covering and $f : Y \rightarrow \text{Alb}(Y)$ is a closed embedding. Since $\chi(Y, K_\phi) = \sum_\nu (-1)^\nu \dim H^\nu(Y, K_\phi)$, for $\gamma \in \Gamma$ the trace $\text{tr}(\gamma) = \sum_\nu (-1)^\nu \text{tr}(\gamma; H^\nu(\tilde{Y}, \delta_{\tilde{Y}}))$ can be written

$$\text{tr}(\gamma) = \sum_\phi \chi(Y, K_\phi) \cdot \text{tr}(\gamma; V_\phi) .$$

K_ϕ has rank $\dim(V_\phi)$ on U , thus generic rank $\dim(V_\phi)$ on Y . Hence the characteristic cycle of the D-module on $\text{Alb}(Y)$ attached to $f_*(K_\phi)$ is a sum of irreducible Lagrangian cycles containing the conormal Lagrangian cycle $\Lambda_{f(Y)} \subset T^*(X)$ with multiplicity $\dim(V_\phi)$. As shown in [FK], by the theorem of Dubson-Riemann-Roch this implies $\chi(K_\phi) = \chi(f_*(K_\phi)) \geq \dim(V_\phi) \cdot \deg(\Lambda_{f(Y)})$. Furthermore since $f(Y) \cong Y$ is smooth, the characteristic variety of $f(Y)$ is $\Lambda_{f(Y)}$ and hence $\deg(\Lambda_{f(Y)})$ is the Euler-Poincare characteristic χ_Y of the variety Y , again by [FK]. This implies

$$\text{tr}(\gamma) = \sum_\nu (\dim(V_\phi)\chi_Y + a_\phi) \cdot \phi(\gamma)$$

for certain integers $a_\phi \geq 0$. Hence the virtual representation defined by $\text{tr}(\gamma)$ is χ_Y times the regular representation of Γ plus a true representation of Γ . Notice that $\chi_Y \geq 0$ holds by [FK] and our assumptions on f .

Remark. In the case of surfaces Y , for a nontrivial irreducible representation ϕ of Γ from this Chevalley-Weil type trace formula we obtain the estimate $m_0(\phi) - 2m_1(\phi) = \dim(V_\phi)\chi_Y + a_\phi \geq 0$. So, for surfaces and nontrivial ϕ under the assumptions before corollary 2, this improves the previous estimate $m_0(\phi) \geq 2m_1(\phi)/(g+1)$ of corollary 2.

References:

- [BBD] Beilinson A., Bernstein J., Deligne P., *Faisceaux pervers*, Asterisque 100 (1982).
- [FK] Franecki J., Kapranov M., *The Gauss map and a noncompact Riemann-Roch formula for constructible sheaves on semiabelian varieties*, Duke Math. J. 104 no. 1 (2000) 171-180.
- [KiW] Kiehl R., Weissauer R., *Weil conjectures, Perverse Sheaves and l-adic Fourier Transform*, Springer Verlag, Ergebnisse der Mathematik 42 (2001).
- [KrW] Krämer Th., Weissauer R., *Vanishing theorems for constructible sheaves on abelian varieties*, J. Alg. Geom. 24 (2015), 531 - 568.
- [W] Weissauer R., *Vanishing theorems for constructible sheaves on abelian varieties over finite fields*, To appear in Math. Annalen.